Algebraic Topology Semestral Examination Back Paper, 2006, M.Math 1st Year

Attempt any four questions. Each question carries 25 marks. You may consult books and notes.

1. (i): Let $C_1 \hookrightarrow C_0$ and $D_1 \hookrightarrow D_0$ be four abelian groups. Show that:

$$(C_0/C_1) \otimes (D_0/D_1) \simeq \frac{C_0 \otimes D_0}{C_1 \otimes D_0 + C_0 \otimes D_1}$$

   (*Hint:* View $C_. := 0 \to 0.. \to C_1 \to C_0$ and $D_. := 0 \to 0... \to D_1 \to D_0$ as two-term chain complexes and apply Kunneth formula to the tensor product chain complex $C_. \otimes D_.$)

   (ii): Show that $\mathbb{RP}(2)$ is not a retract of $\mathbb{RP}(3)$.

2. (i): Prove that S^1 is not a covering space of the bouquet of 2 circles $S^1 \vee S^1$.

   (ii): Compute $\hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$.

3. (i): Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a continuous map, where $\mathbb{T}^2 = S^1 \times S^1$. Prove that $f_* : H_1(\mathbb{T}^2, \mathbb{Z}) \to H_1(\mathbb{T}^2, \mathbb{Z})$ is an isomorphism iff $f_* : H_2(\mathbb{T}^2, \mathbb{Z}) \to H_2(\mathbb{T}^2, \mathbb{Z})$ is an isomorphism. (*Hint:* Use the $\mathbb{Z}$-cohomology ring of $\mathbb{T}^2$.)

   (ii): Let $f : S^2 \to \mathbb{T}^2$ be a continuous map. Show that $f_* : H_i(S^2, \mathbb{Z}) \to H_i(\mathbb{T}^2, \mathbb{Z})$ is the zero homomorphism for $i = 1, 2$.

4. (i): Let $f_n : S^1 \to S^1$ be the map $z \to z^n$ $(n \in \mathbb{N})$. Show that the topological mapping cone $C(f_n)$ is homotopically equivalent to $C(f_m)$ iff $m = n$.

   (ii): Let M be a compact connected orientable manifold of dimension n. Let $\alpha \in H^i(M, \mathbb{Z})$ (where $0 \le i \le n$) be a cohomology class such that $\alpha \cup \beta = 0$ for all $\beta \in H^{n-i}(M, \mathbb{Z})$. Show that $\alpha = 0$.

5. (i): Compute $H_3(S^2 \times \mathbb{RP}(2), \mathbb{Z})$.

   (ii): Prove that the map $f : \mathbb{RP}(2) \to \mathbb{RP}(2)$ defined by $[x_0 : x_1 : x_2] \mapsto [x_0 + 2x_1 - x_2 : x_1 - 3x_2 : x_2]$ does not lift to a map $\widetilde{f} : \mathbb{RP}(2) \to S^2$, where $\pi : S^2 \to \mathbb{RP}(2)$ is the usual covering projection.